Lecture 10

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MMS

aka Maximin Share

So far we've seen: EF & PROP

EF $\downarrow$ depends on what others get — "comparison-based"

PROP $\downarrow$ only depends on what I get — "share-based"

MMS roughly extends proportionality to indivisible goods.

Will focus on additive goods for this lecture.

MMS guarantee: Given an instance $(N, M, (v_i)_{i \in N})$, $n = |N|$, $m = |M|$

Let $\Pi^{(n)}$ be the set of all n-partitions of M.

The agent i's MMS guarantee

$$\boxed{\mu_i^{(n)} = \max_{P \in \Pi^{(n)}} \min_{j \in [n]} v_i(P_j)}$$

That is, agent i gets to choose an n-partition of goods, & she gets the worst bundle in that partition.

Eg. Say 3 agents, 6 goods, $v_1 = (3, 6, 12, 9, 2, 5)$

Then A is an MMS allocation if $\forall i$, $v_i(A_i) \geq \mu_i^{(n)}$.

Eg.

	1	2	3	4	5	6	
v_1	3	5	1	7	6	4	$\to$ 13
v_2	8	8	12	4	9	7	$\to$ 24

Easy to see that $\mu_i^{(n)} \leq v_i(M)/n$

(thus a proportional allocation is MMS)

A partition $P = (P_1, \dots, P_n)$ of goods M is an MMS-witness for agent i, if $\forall j \in [n]$, $v_i(P_j) \geq \mu_i^{(n)}(M)$

Lemma 1: The MMS guarantee is NP-hard to compute, even for 2 agents.

Lemma 2: A $(1-\varepsilon)$-approximation to the MMS guarantee can be computed in time poly$(|I|, 1/\varepsilon)$

i.e., can obtain $\hat{\mu}_i^{(n)} \in [(1-\varepsilon)\mu_i^{(n)}, \mu_i^{(n)}]$ in polynomial time.

Claim: For 2 agents MMS $\Rightarrow$ EFX.

Theorem: An MMS allocation may not exist, even for 3 agents, 9 items.

3 agents: R, C, U

		max			max			max
1	1, 1, 1	1	2	16, 16, 15	16	3	23, 22, 23	23
4	26, 26, 25	26	5	4, 4, 4	4	6	10, 9, 10	10
7	12, 13, 13	13	8	19, 20, 20	20	9	9, 9, 9	9

$\mu_R^{(3)} = 40 \quad \mu_C^{(3)} = 40 \quad \mu_U^{(3)} = 40$

Giving each agent a row doesn't work: both C & U want R3.

Giving each agent a column doesn't work: both R & U want C3.

Let $\max_j = \max_i v_{ij}$.

$\sum_{j \in M} \max_j = 40 + 40 + 42 = 122$

So each agent must get 40, 41, or 42. Also for any bundle B, $\sum_{j \in B} \max_j \in \{40, 41, 42\}$

- no single item gives any agent ≥ 40
- only 2 item bundle possible is 2 & 4

  So let $B = \{2, 4\}$, then $\sum_{j \in M \setminus B} \max_j = 80$. Then for the other bundles X, Y value of agent getting the bundles = max value of the bundle = 40

  thus each item in $M \setminus B$ must go to agent w/ value = $\max_j$.

  But then agent R cannot get 5 & 7, must get 3, but no way to get exactly 40.

  Similarly for agent C.
- So all 3 agents must get 3 item bundles.

  Can check that only possible bundles containing 9 are 7 8 9 & 3 6 9. Both have total max value = 42.

  So the two remaining bundles must have max value = agent value = 40.

  Can check we can't get this.

Overall, shows that we can't get better than a $39/40$-approximation to MMS.

Recent work gives a gap of $1/21$ for 4 agents, and more generally a gap of $1 - \Omega\left(\frac{1}{\log^2 n}\right)$

How close can we get to these values?

A 2/3-MMS approximation.

Say we are given an instance $I = (N, M, (v_i)_{i \in N})$.

Note that scaling agent i's valuation by λ_i multiplies the value of any bundle, as well as the MMS guarantee, by λ_i. Hence an α-MMS allocation remains an α-allocation.

Thus for each agent, scale values so that $\mu_i^{(n)} = 1$. So $v_i(M) \geq n$, & for a 2/3-MMS, have to guarantee each agent value 2/3.

(Note that this scaling is NP-hard, since computing $v_i^{(n)}$ is NP-hard, but we can get a $(1-\varepsilon)$-approximation, so overall we will get $2/3(1-\varepsilon)$).

Lemma 1: Let $S \subseteq M$, $k = |S|$. Then $\forall i \in N$, $\mu_i^{(n-k)}(M \setminus S) \geq \mu_i^{(n)}$.

Proof. Let $P = (P_1, \dots, P_n)$ be agent i's MMS partition.

Remove any bundles that intersect w/ S. This removes at most k bundles.

Thus at least $n-k$ bundles remain unaffected. This is a partition of a (subset of) $M \setminus S$ into $n-k$ bundles, each of which has value at least $\mu_i^{(n)}$. Thus $\mu_i^{(n-k)}(M \setminus S) \geq \mu_i^{(n)}$. ∎

Corollary: Given an instance, if give an agent a single item & remove both the item & the agent, the MMS share of the remaining $(n-1)$ agents for the remaining items does not decrease.

Say an instance I is "ordered" if for some ordering $g_1, \dots, g_m$ of the goods, $\forall i$, $v_i(g_1) \geq v_i(g_2) \geq \dots \geq v_i(g_m)$.

Lemma 2: For any instance I w/ n agents, m goods, $\exists$ an ordered instance w/ n agents, m goods s.t. that:

1. MMS share values are unchanged:

   $\mu_i^{(n)}(I) = \mu_i^{(n)}(I') \quad \forall i$
2. Any allocation $A' = (A_1', \dots, A_n')$ in I' can be converted efficiently into an allocation $A = (A_1, \dots, A_n)$ in I so that $\forall i$, $v_i(A_i) \geq v_i'(A_i')$

Corollary: Given an algorithm that obtains α-MMS on ordered instances, we can get an algorithm that obtains α-MMS for all instances w/ additive goods.

Proof of Lemma: Given an instance I, consider the instance w/ goods $g_1', \dots, g_m'$ where $v_i(g_j') =$ value of i's j^{th} highest good in I.

Clearly: $v_i(g_1') \geq v_i(g_2') \geq \dots \geq v_i(g_m') \quad \forall i$

Also $\forall i$, the multiset $\{v_i(g_j')\}_{j \in [m]} = \{v_i(g_j)\}_{j \in [m]}$

Thus, $\forall i$, $\mu_i^{(n)}(I) = \mu_i^{(n)}(I')$.

Now say we are given an allocation $A' = (A_1', A_2', \dots, A_n')$ in I'.

To get A, we implement the following picking sequence:

- If g_j' is assigned to i, then i appears in posn. j in the picking sequence.

In A': 3 3 2 1 1 1 2 1

Eg. g_1' g_2' g_3' g_4' g_5' g_6' g_7' g_8'

The agents in I pick goods in this picking sequence.

Now suppose $A_i' = \{g_{i_1}', g_{i_2}', \dots, g_{i_r}'\}$ where $i_1 > i_2 > \dots > i_r$.

So i appears at posns. $i_1, i_2, \dots, i_r$ in the picking sequence.

When she picks first, at most $(i_1 - 1)$ goods have been chosen. So she can still pick her i_1^{th} most valuable good.

So for each i_j, the good she picks has value at least $v_i(g_{i_j}')$.

Thus $v_i(A_i) \geq v_i(A_i')$ ∎

Algorithm: Assume I is an ordered instance.

For convenience, assume throughout that $n' = |N'|$, $m' = |M'|$.

Will maintain throughout that if N' is set of remaining agents, M' is set of remaining goods, $n' = |N'|$, then $\mu_i^{(n')}(M') \geq 1$.

Let $N' = N$, $M' = M$, $n' = |N'|$

Also $m' \geq n' + 1$, else we just give each agent a single good, this is an MMS allocation.

1. while $\exists i \in N'$ s.t. $v_i(g_1) \geq 2/3$

   $A_i \leftarrow \{g_1\}$, $N' \leftarrow N' \setminus i$, $M' \leftarrow M' \setminus g_1$

By Lemma 1, the MMS value of the remaining agents does not decrease. Hence $\mu_i^{(n')}(M') \geq 1$.

2. while $\exists i \in N'$ s.t. $v_i(g_{n'}) + v_i(g_{n'+1}) \geq 2/3$,

   $A_i \leftarrow \{g_{n'}, g_{n'+1}\}$, $N' \leftarrow N \setminus i$, $M' \leftarrow M' \setminus \{g_{n'}, g_{n'+1}\}$

Claim: MMS values of remaining agents does not decrease.

Proof: Let N' be agents before i is removed, $n' = |N'|$, $M' = M' \cup \{g_{n'}, g_{n'+1}\}$

Fix an agent k. Let $P = (P_1, \dots, P_{n'})$ be a partition of M' that is an MMS witness for k

(i.e. $\forall j \in [n']$, $v_k(P_j) \geq \mu_i^{n'}(M')$).

Case I: if $g_{n'}, g_{n'+1}$ are in the same bundle.

Simply remove that bundle. Then there are $n'-1$ remaining bundles, each of value at least $\mu_i^{(n')}(M')$.

Case II: If $g_{n'}, g_{n'+1}$ are in diff bundles, say $P_{n'-1}$ & $P_{n'}$.

At least one of these two bundles contains a good from $g_1, \dots, g_{n'-1}$, say $P_{n'-1}$.

Then the bundle $P_{n'} \cup P_{n'-1} \setminus \{g_n, g_{n+1}\}$ has value at least $v_i(P_n) - v_i(g_n) + v_i(g_{n-1}) \geq v_i(P_n)$.

Thus we get $n'-1$ bundles each of value at least $\mu_i^{(n')}(M')$. ∎

3. Now we have N', M' so that: $\forall i$, $v_i(g_1) < 2/3$

   & $v_i(g_{n'}) + v_i(g_{n'+1}) < 2/3$.

   (thus $\forall j \geq n'+1$, $v_i(g_j) \leq 1/3 \quad \forall i$).

   & $\forall i \in N'$, $\mu_i^{(n')}(M') \geq 1$.

   Initially, consider the partition:

   $P_1 = \{g_1\}, P_2 = \{g_2\}, \dots, P_{n'-1} = \{g_{n'-1}\}, P_{n'} = \{g_{n'}, g_{n'+1}\}$

   Call the remaining items "small" items. (and note that $v_i(g) \leq 1/3$ for small items).

   while $\exists i \in N'$:

   Pick an unallocated bundle, say P_j.

   Add small goods to P_j until for some agent $i \in N'$, $v_i(P_j) \geq 2/3$.

   $A_i \leftarrow P_j$, $N' \leftarrow N' \setminus i$, $M' \leftarrow M' \setminus P_j$.

   Allocate remaining goods arbitrarily.

Claim: Before any iteration of the while loop, $\forall i \in N'$, $v_i(M') \geq n'$.

(prove yourself).

Best known approximation is $7/9$.